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Geometry, Complexity, and Combinatorics of Permutation Polytopes*

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Each group G of permutation matrices gives rise to a *permutation polytope* $P(G) = \text{conv}(G) \subset \mathbb{R}^{d \times d}$, and for any $x \in \mathbb{R}^d$, an *orbit polytope* $P(G, x) = \text{conv}(G \cdot x)$. A broad subclass is formed by the *Young permutation polytopes*, which correspond bijectively to partitions $\lambda = (\lambda_1, ..., \lambda_k) \vdash n$ of positive integers, and arise from the *Young* representations of the symmetric group. Young polytopes provide a framework allowing a unified study of many combinatorial optimization problems of different computational complexities. In particular, the much studied *traveling salesman polytope* is a certain Young orbit polytope, and many decision problems, such as simplical complex isomorphism, reduce to optimizing linear functionals over Young polytopes. First, the classical polytope of bistochastic matrices $P(S_n) = P((n-1, 1))$ is studied. Large stable sets in its 1-skeleton, induced by the Young representations, are exhibited, and it is shown that its stability number $\alpha(n)$ is $2^{\Omega(\sqrt{n}\log n)}$. Next, we study low dimensional skeletons of Young polytopes in general. Letting m be the largest integer for which $P(\lambda)$ is m-neighborly, under some restrictions on λ it is shown that $\lfloor k^2/2 \rfloor \leqslant m < \frac{1}{2}(k+1)!$. Finally, we study the following semialgebraic geometric question, posed by D. Kozen: Is the combinatorial type of the polytope, and oriented matroid, of a generic orbit, unique? We show that, while a theorem of Rado implies a positive answer for the symmetric group, the general answer is negative, and the induced stratifications are nontrivial, and should be the subject of a future study.

1. INTRODUCTION

Let $\pi: S_d \to GL(\mathbb{R}^d)$ be the *standard representation* of the symmetric group on d elements assigning to each $\sigma \in S_d$ the corresponding permutation matrix with respect to the standard basis of $\mathbb{R}^d$. For a subgroup G of S_d, we define the *permutation polytope* of G to be $P(G) = \text{conv}(\{\pi(\sigma): \sigma \in G\})$, and for $x \in \mathbb{R}^d$, the *orbit polytope* of x to be the convex

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hull of its orbit $P(G, x) = \mathrm{conv}(\{\pi(\sigma)(x) : \sigma \in G\})$, which is a projection of $P(G)$.

In the case $G = S_d$, both the permutation polytope and the orbit polytopes are well understood. Hardy, Littlewood, and Pólya [11] and Birkhoff [6] have shown that the so-called *assignment polytope* $P(S_d)$ is exactly the set of bistochastic d by d matrices, whereas Rado [22], in the context of inequalities, essentially determined the combinatorial structure of the so-called *permutohedron* $P(S_d, x)$.

Little however is known in general, though already in [22] the orbit polytopes were defined for an arbitrary subgroup, and Mirsky [17] posed the question about the general permutation polytope, indicating that in general the problem (of describing the facets of these polytopes) "turns out to be unexpectedly difficult."

As was observed by A. I. Barvinok and A. M. Vershik [3], permutation polytopes are closely related to the study of many combinatorial optimization problems. On one hand, this relation provides evidence that the combinatorial structure of these polytopes is probably intractable in general, and justifies Mirsky's claim about the difficulty of investigating them. On the other hand, studying them in a unified way might illuminate computational complexity aspects of combinatorial optimization.

It should be noted that permutation polytopes are $\{0, 1\}$-polytopes, and as such, possess some special properties. In particular, Naddef and Pulleyblank [19], generalizing results of Brualdy and Gibson for the assignment polytope [9], showed that the 1-skeleton of such a polytope is always either a hypercube or Hamilton connected. Also, the diameter of such a polytope is at most its dimension (see [18] for a proof and [14] for a generalization). At this point, however, the problem of efficiently characterising the 1-skeletons of $\{0, 1\}$-polytopes, raised in [19], is yet unsettled.

In the next section we recall the definition of the class of Young representations of the symmetric group from the combinatorial point of view taken in [12]. With any partition $\lambda = (\lambda_1, ..., \lambda_k) \vdash n$ into k positive parts, there is associated a *λ-Young representation* of the symmetric group, which turns S_n into a subgroup S_n^* of S_{d_λ}, where $d_\lambda = n!/\prod_{i=1}^{k} \lambda_i!$. We refer to the corresponding permutation polytope $P(S_n^*)$ as the *λ-Young polytope*, and denote it by $P(\lambda)$, and a corresponding orbit polytope is denoted by $P(\lambda, x)$. Also in Section 2, following [32], we demonstrate how many interesting combinatorial optimization problems can be embedded in this framework. For example, the problem of deciding the isomorphism of two k-uniform hypergraphs with n vertices is reduced to the problem of optimizing a linear functional over the Young polytope $P((n-k, k))$. Another example is provided by the *symmetric traveling salesman polytope*, which is an orbit polytope $P((n-2, 2), h_n)$ for an appropriate $h_n \in \mathbb{R}^{\binom{n}{2}}$.

In Section 3 we investigate the assignment polytope, the study of which goes back to the twenties [11]. In particular, its 1-skeleton had been extensively studied; see, for example, [1] and the second of a series of three papers devoted to the assignment polytope [9], where many interesting properties were established.

Viewing the assignment polytope $P(S_n) = P((n-1, 1))$ in the framework of Young polytopes, we show that for any nontrivial partition $\lambda \vdash n$, the subgroup S_n^* gives rise to a stable set of vertices in the 1-skeleton of $P(S_{d_\lambda})$. In particular, the stability number $\alpha(n) = \alpha(P(S_n))$ is $2^{\Omega(\sqrt{n}\log n)}$, which somewhat contrasts its strong connectivity properties (its diameter is at most 2 [1] and it is Hamilton connected [19]). We also discuss the computational complexity of recognizing these large stable sets.

In contrast, for any partition λ as above, the Young polytope $P(\lambda)$ itself turns out to be 2-neighborly. In Section 4 we study the neighborliness degree of Young polytopes. Letting l be the largest integer for which $P(\lambda)$ is l-neighborly, it is shown that under some restrictions on a partition λ with k positive parts, $\lfloor k^2/2 \rfloor \leqslant l < \frac{1}{2}(k+1)!$. A sufficient condition for $P(\lambda)$ to be l-neighborly is derived and reduced to a graph theoretical statement on vertex coloring, and computational complexity aspects are discussed. It is interesting, in the context of path following algorithms for optimizing linear functionals over convex polytopes, that the associated decision problem on the sequence $P((n-2, 2))$ is NP-complete, while the graph of $P((n-2, 2))$ has diameter 1. It is also interesting that, while the adjacency relation of the sequence $P((n-2, 2))$ is trivial, it is known to be NP-complete for the projected sequence $P((n-2, 2), h_n)$ of traveling salesman polytopes [21].

Given a subgroup G of S_d, the orbit polytopes form a polytope bundle $\{P(G, x) : x \in \mathbb{R}^d\}$ (cf. [5]), and their combinatorial type induces the *polytope stratification* of the base space $\mathbb{R}^d$. A sequence of partitions, such as $(n-2, 2)$, $n \geqslant 4$, yields a sequence $\{P((n-2, 2), x) : x \in \mathbb{R}^{\binom{n}{2}}\}$ of polytope bundles, and for each specified sequence of vectors $x_n \in \mathbb{R}^{\binom{n}{2}}$ we obtain a sequence of polytope fibers $P((n-2, 2), x_n)$, on which the linear optimization-related decision problem is considered. It turns out that for some $\{0, 1\}$-valued sequences, the decision problem belongs to the computational complexity class P, while in other cases it is NP-complete. This observation, made in [3], motivates the study of the polytope stratification. A first step is to restrict attention to orbits of generic points, i.e., points the coordinates of which are algebraically independent over the rationals. Indeed, Rado's work [22] implies that for any generic point, $P(S_d, x)$ is the Permutohedron. This naturally leads to the following question suggested by Dexter Kozen. Is it true that, for an arbitrary subgroup G of S_d, the combinatorial type of the orbit polytope is the same for any generic point?

In Section 5 we show that the answer to this question is negative. We study the polytope stratification of $\mathbb{R}^d$ and the related stratifications induced by the affine matroid and oriented matroid of an orbit. For each subgroup G, a generic matroid $M(G)$ is defined, and it is shown that, while any two generic points have the same orbit matroid $M(G)$, different generic points may have nonisomorphic orbit polytopes. This shows that the polytope stratification is nontrivial even on the set of generic points, and thus provides a negative answer to Kozen's question. In our construction, G is the $(2, 2)$-Young representation of S_4 in $GL(\mathbb{R}^6)$, and we establish the existence of generic points $x, y \in \mathbb{R}^6$ for which the polytopes $P(G, x)$ and $P(G, y)$ have different number of faces of the same dimension.

2. Young Representations and Combinatorial Optimization

We start by establishing some notation. Given a finite set A, we denote by S_A the group of permutations of elements of A, and by $\mathbb{R}^A$ the vector space of real valued functions on A, with the standard basis $\{e_a : a \in A\}$ of Kronecker functions, $e_a(b) = \delta_{a,b}$ $(a, b \in A)$. Similarly, we denote by $\operatorname{mat}(A, \mathbb{R})$ the algebra of real matrices with the standard basis $\{e_{a,b} : a, b \in A\}$. Given a point $x \in \mathbb{R}^A$, its *support* is $\operatorname{supp}(x) = \{a : x_a \neq 0\} \subseteq A$, and for a point $x \in \operatorname{mat}(A, \mathbb{R})$, let $\operatorname{supp}(x) = \{(a, b) : x_{(a,b)} \neq 0\} \subseteq A \times A$. For a subset X of any real space, we let $\operatorname{supp}(X) = \bigcup \{\operatorname{supp}(x) : x \in X\}$. We can identify, by means of the standard basis, the algebra $L(\mathbb{R}^A)$ of linear transformations from $\mathbb{R}^A$ to itself, with $\operatorname{mat}(A, \mathbb{R})$. These two are also identified with the tensor product $\mathbb{R}^A \otimes \mathbb{R}^A$, by sending a matrix $M \in \operatorname{mat}(A, \mathbb{R})$ to the tensor $\sum_{a,b \in A} M_{a,b} e_a \otimes e_b$. The standard representation of S_A is the group homomorphism $\pi : S_A \to GL(\mathbb{R}^A) \subseteq L(\mathbb{R}^A)$ given by $\pi(\sigma)(e_a) = e_{\sigma(a)}$ $(a \in A)$. When $A = [d] = \{1, ..., d\}$, we use the abbreviated notation S_d, $\mathbb{R}^d$, and $\operatorname{mat}(d, \mathbb{R})$.

We now describe the class of Young representations of the symmetric group. For a thorough discussion of this theory the reader is referred to [12]. We remark that this class is complete in the sense that the characters of Young representations form a $\mathbb{Z}$-basis for the lattice of characters of the symmetric group.

Given a partition $\lambda = (\lambda_1, ..., \lambda_k)$ of $n \in \mathbb{N}$ into k positive integers $\lambda_1 \geqslant \cdots \geqslant \lambda_k$, a *$\lambda$-diagram* is a set $D = \{(i, j) : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \lambda_i\}$. A *$\lambda$-tableau* is a bijection from D onto $[n]$. Thinking of a pair $(i, j) \in D$ as the (i, j)th entry of an n by n matrix, we arrive at the more pictorial way, attributed to A. Young, of viewing a λ-diagram as a table having k left justified rows, where the ith row has λ_i entries. A λ-tableau is then a

λ-diagram filled with the numbers $1, ..., n$, each appearing exactly once. Define now an equivalence relation on λ-tableaux, where two tableaux are related if they have the same set of numbers on each row. An equivalence class t is called a *λ-tabloid*. When dealing with partitions with large number k of parts, it will be convenient to think of a tabloid t simply as an ordered list of subsets forming a partition of $[n]$, where the ith subset, denoted by $t[i]$, is the λ_i-set (set of size λ_i) constituting of the ith row. For example, $t = (\{1, 3, 6\}, \{2, 4\}, \{5, 7\})$ is a $(3, 2, 2)$-tabloid. The permutation group S_n acts naturally on λ-tableaux, where a permutation $\sigma \in S_n$ sends a given tableau to the one obtained from it by applying σ to its entries. This action clearly induces a well defined action on the set $T(\lambda)$ of λ-tabloids. When the partition λ is clear from the context, we will denote by σ^* the permutation on $T(\lambda)$ induced from $\sigma \in S_n$. Thus, $*: S_n \to S_{T(\lambda)}$ is a group homomorphism, and the image S_n^* of S_n is a subgroup of $S_{T(\lambda)}$. Note that the number of λ-tabloids is $\binom{n}{\lambda} = (\sum_{i=1}^{k} \lambda_i)!/\prod_{i=1}^{k} \lambda_i!$.

Now, let $\{e_t : t \in T(\lambda)\}$ be the standard basis of Kronecker functions of the space $\mathbb{R}^{T(\lambda)}$. The *λ-Young representation* of the symmetric group, $\pi_\lambda : S_n \to GL(\mathbb{R}^{T(\lambda)})$, is defined by $\pi_\lambda(\sigma)(e_t) = e_{\sigma^*(t)}$ for all $\sigma \in S_n$, $t \in T(\lambda)$. We then have the *Young permutation polytope*

$$\begin{aligned} P(\lambda) &= \operatorname{conv}(\{\pi_\lambda(\sigma) : \sigma \in S_n\}) = \operatorname{conv}(\{\pi(\sigma^*) : \sigma^* \in S_n^*\}) \\ &= P(S_n^*) \subseteq \operatorname{mat}(T(\lambda), \mathbb{R}), \end{aligned}$$

and, for $x \in \mathbb{R}^{T(\lambda)}$, the *Young orbit polytope*

$$\begin{aligned} P(\lambda, x) &= \operatorname{conv}(\{\pi_\lambda(\sigma)(x) : \sigma \in S_n\}) = \operatorname{conv}(\{\pi(\sigma^*)(x) : \sigma^* \in S_n^*\}) \\ &= P(S_n^*, x) \subseteq \mathbb{R}^{T(\lambda)}. \end{aligned}$$

EXAMPLE 2.1. Let $\lambda = (n-1, 1)$. Then, identifying a tabloid $t \in T(\lambda)$ with the number $i \in [n]$ such that $t[2] = \{i\}$, we can identify in that case S_n^* with S_n, so $P((n-1, 1)) = P(S_n)$ is the assignment polytope.

We now demonstrate the connection between Young polytopes and combinatorial optimization. The discussion below is similar to [3, 2] and is included for the completeness of the exposition. Consider a sequence of rational polytopes $P_n \subseteq \mathbb{R}^{d(n)}$, $n \in \mathbb{N}$, specified in some uniform way. The *optimization problem* on this sequence is the problem of obtaining, given $n \in \mathbb{N}$ and a rational vector $c_n \in \mathbb{Q}^{d(n)}$, the value $z = \max\{\langle c_n, x \rangle : x \in P_n\}$. The *decision problem* on the sequence is, given $n \in \mathbb{N}$ and $c_n \in \mathbb{Q}^{d(n)}$ as above, and a rational number $q \in \mathbb{Q}$, to decide if $z \geqslant q$. The size of an input instance (n, c_n, q) will be taken as $n + \operatorname{size}(c_n) + \operatorname{size}(q)$ (cf. [23]).

Now, any $k-1$ positive integers $\lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_k$ yield a sequence

$$\lambda(n) = \left(n - \sum_{i=2}^{k} \lambda_i, \lambda_2, ..., \lambda_k\right), \qquad n \geqslant \lambda_2 + \sum_{i=2}^{k} \lambda_i$$

of partitions of n, and this yields a sequence of permutation polytopes $P(\lambda(n))$. If, in addition, a sequence $x_n \in \mathbb{R}^{T(\lambda(n))}$ of vectors is specified, then we also have a sequence of orbit polytopes $P(\lambda(n), x_n)$. We remark that this way of producing a sequence of partitions appears implicitly in many contexts, e.g., in [13], where the asymptotics of the rank function of a compatible series of symmetric matroids on $T((n-k, k))$ is addressed.

EXAMPLE 2.2. Let k be a positive integer and $\lambda(n) = (n-k, k)$. A tabloid t can then be identified with the k-set, or hyperedge $t[2]$ of the complete k-uniform hypergraph on the set of vertices $[n]$. Thus, a vector $x \in \mathbb{R}^{T((n-k,k))}$ could be identified with a weighted k-uniform hypergraph, having set of hyperedges $\operatorname{supp}(x)$, where the weight of an edge $t \in \operatorname{supp}(x)$ is x_t. If s is $\{0, 1\}$-valued, then x is an (unweighted) k-uniform hypergraph. If $k=2$, it is simply an abstract graph. Given two k-uniform hypergraphs $h, g \in \mathbb{R}^{T((n-k,k))}$, having $q = |\operatorname{supp}(h)| = |\operatorname{supp}(g)|$ hyperedges each, and letting $c = h \otimes g$, the decision problem on $P(\lambda(n))$ with input (n, c, q) is exactly the decision question of whether or not g and h are isomorphic as hypergraphs. Note also that the collection of maximal simplices of a $(k-1)$-dimensional pure simplicial complex is a k-uniform hypergraph, so the isomorphism of pure simplicial complexes could be decided in the same way.

EXAMPLE 2.3. Let $h_n \in \mathbb{R}^{T((n-2,2))}$ be a graph which is an n-cycle. For instance, let $n=5$, let $t_1 = (\{3, 4, 5\}, \{1, 2\}), ..., t_4 = (\{1, 2, 3\}, \{4, 5\})$, and $t_5 = (\{2, 3, 4\}, \{5, 1\})$. Then h_5 could be taken as $h_5 = \sum_{i=1}^{5} e_{t_i}$. The sequence of orbit polytopes $P((n-2, 2), h_n)$, $n \in \mathbb{N}$, is the sequence of *symmetric traveling salesman polytopes*, for which the decision problem is NP-complete. More precisely, given a graph $c \in \mathbb{R}^{T((n-2,2))}$, the decision problem on $P((n-2, 2), h_n)$ with input (n, c, n) is exactly the decision problem of whether or not c has a Hamiltonian cycle.

Similarly, taking $k(n) \leqslant n$ and a complement of a $k(n)$-clique $s_n = K_n \backslash K_{k(n)}$, the decision problem on $P((n-2, 2), s_n)$ with input (n, c, q), where $c \in \mathbb{R}^{T((n-2,2))}$ is a graph and $q = |\operatorname{supp}(c)|$, is the decision problem of wheter or not c has a stable set of size at least $k(n)$.

For some nontrivial orbits, however, the optimization and decision problems can be solved in polynomial time. Let m_n be an $\lfloor n/2 \rfloor$-matching. Then, the solution for the optimization problem on $P((n-2, 2), m_n)$ with input (n, c), where $c \in \mathbb{R}^{T((n-2,2))}$, is the maximum weight of a matching in the weighted graph c.

Thus, as observed in [3], the decision problem on the orbit polytopes is NP-complete in general. Observing that for any matrix $A \in \mathrm{mat}(d, \mathbb{R})$ and vectors $x, c \in \mathbb{R}^d$ we have $\langle c, Ax \rangle = \langle c \otimes x, A \rangle$, it is clear that the optimization problem for any orbit polytope $P(G, x)$ on input c, could be solved by solving the optimization problem on the corresponding permutation polytope $P(G)$ on input $c \otimes x$. Thus, the combinatorial structure of Young polytopes and their orbit counterparts is probably intractable in general. A more direct evidence to this statement was given in [21], where it was proved that the adjacency relation on the class $P((n-2, 2), h_n)$ of symmetric traveling salesman polytopes is NP-complete. Nevertheless, in Section 4 we establish some statements on the combinatorial structure of Young polytopes.

3. On the Graph of the Assignment Polytope

Given a convex polytope P with a set of vertices $V = \mathrm{ext}(P)$, we will refer to a subset $F \subseteq V$ as a *face* of P if for some face G of P we have $F = G \cap V$. The 1-*skeleton,* or *graph,* of P, is the abstract graph on V in which the edges are the 1-faces of P. A subset $S \subseteq V$ is *stable* in P if it is stable in the graph of P, i.e., no two vertices $u, v \in S$ form an edge (1-face) of P.

In this section, we study the assignment polytope $P(S_n)$, and show that subgroups induced by Young representations give rise to large stable sets of vertices in its 1-skeleton. In particular, letting $\alpha(n) = \alpha(P(S_n))$ be the *stability number* of $P(S_n)$, i.e., the largest size of a stable set of $P(S_n)$, we show that $\alpha(n) = 2^{\Omega(\sqrt{n \log n})}$. This is somewhat surprising, since the graph of $P(S_n)$ is Hamilton connected [19] and its diameter is 2 for all $n \geq 4$ [1].

Before going on, we recall some definitions and properties of convex polytopes and oriented matroids. For the theory of convex polytopes, consult, for example, [10, 8], and for oriented matroids [7].

A convex polytope P is *k-neighborly* if every k-subset of its vertices is a $(k-1)$-face of P. If P is k-neighborly, then it is i-neighborly for $i = 0, 1, \ldots, k$. We define the *neighborliness degree* of P to be the largest k for which it is k-neighborly.

Given a set of points in affine space, $V \subseteq \mathbb{R}^d$, a *Radon partition* of V is a partition (S, T) of V such that $\mathrm{conv}(S) \cap \mathrm{conv}(T) \neq \emptyset$. A pair (S, T) of sets of points in $\mathbb{R}^d$ is a *minimal Radon partition* if it is a Radon partition of $S \cup T$, and no proper subset of $S \cup T$ admits a Radon partition. Note that, in the language of oriented matroids (cf. [7]), given a finite set $V \subseteq \mathbb{R}^d$ and a pair (S, T) of subsets of V, the pair is a minimal Radon partition exactly when it is an oriented circuit of the affine oriented matroid on V. We will therefore call a pair (S, T) an *oriented circuit* of V if (S, T) is a minimal Radon partition and $S, T \subseteq V$.

We need the following statement from [16] (see also [7, Proposition 9.1.2]).

PROPOSITION 3.1. *Let $P \subseteq \mathbb{R}^d$ be a convex polytope and $V = \operatorname{ext}(P)$. A subset $F \subseteq V$ is not a face of P if and only if there exists an oriented circuit (S, T) of V such that $S \subseteq F$ and $T \nsubseteq F$.*

For polytopes contained in the nonnegative orthant $\mathbb{R}^d_+$, we can deduce the following sufficient condition for being a face.

PROPOSITION 3.2. *Let $P \subseteq \mathbb{R}^d_+$ be a convex polytope, $V = \operatorname{ext}(P)$, and $F \subseteq V$. If for all $v \in V \backslash F$ we have* $\operatorname{supp}(v) \nsubseteq \operatorname{supp}(F)$, *then F is a face of P.*

Proof. Let $F \subseteq V$ satisfy the hypothesis of the proposition. Assume indirectly that it is not a face of P, and let (S, T) be an oriented circuit of V as is guaranteed by Proposition 3.1. Now, (S, T) is a Radon partition, so there exists a point $x \in \operatorname{conv}(S) \cap \operatorname{conv}(T)$. Writing x as a convex combination of elements of T, the coefficient of each $t \in T$ is positive, since (S, T) is a minimal Radon partition. But $T \subseteq V \subseteq \mathbb{R}^d_+$, so $\operatorname{supp}(T) = \operatorname{supp}(x) \subseteq \operatorname{supp}(S) \subseteq \operatorname{supp}(F)$, which by hypothesis implies $T \subseteq F$, a contradiction. ▮

Next, we introduce some notation. We denote by e the identity element in S_n. By a *proper cycle* of a permutation, we mean a cycle of length at least two. Given a permutation $\sigma \in S_d$, we denote by $\bar{\sigma} = \pi(\sigma)$ the corresponding matrix assigned to it by the standard representation, regarded as a point in affine space $\operatorname{mat}(d, \mathbb{R})$, and by $\bar{\sigma}_{i,j}$ its (i, j)th entry $(i, j \in [d])$. Similarly, when $\sigma \in S_n$ and $\sigma^* \in S_{T(\lambda)}$ is the induced permutation, where the partition $\lambda \vdash n$ is understood, we have $\bar{\sigma}^* \in \operatorname{mat}(T(\lambda), \mathbb{R})$, and for tabloids $s, t \in T(\lambda)$, its (s, t)th entry is $\bar{\sigma}^*_{s,t}$.

We now turn to discuss the graph of the assignment polytope. Recall the following proposition from [20], which can be derived from Proposition 3.2.

PROPOSITION 3.3. *A 2-subset $F = \{\bar{\sigma}, \bar{\tau}\} \subseteq \{\bar{\sigma} : \sigma \in S_n\} = \operatorname{ext}(P(S_n))$ is an edge of $P(S_n)$ if and only if $\sigma^{-1}\tau$ has exactly one proper cycle.*

By means of the identification given in Example 2.1, we have $P((n-1, 1)) = P(S_n)$, and so Proposition 3.3 implies that, for $n \geqslant 4$, the permutation polytope $P((n-1, 1))$ is not 2-neighborly. It will become evident in the next section that, in contrast, for all $\lambda \vdash n$ other than (n), $(n-1, 1)$, the Young polytop $P(\lambda)$ is 2-neighborly, that is, any two vertices in $V(\lambda) = \{\bar{\sigma}^* : \sigma \in S_n\} = \operatorname{ext}(P(\lambda))$ form an edge of $P(\lambda)$. It is interesting to note that, in contrast yet to this last statement, no two vertices in $V(\lambda)$

form an edge in the assignment polytope $P(S_{T(\lambda)})$. This is the content of the following.

THEOREM 3.4. *Let $\lambda \vdash n$ be any partition other than (n), $(n-1, 1)$. The subset*

$$V(\lambda) = \{\bar{\sigma}^*: \sigma \in S_n\} \subseteq \{\bar{\sigma}: \sigma \in S_{T(\lambda)}\} = \mathrm{ext}(P(S_{T(\lambda)}))$$

is stable in $P(S_{T(\lambda)})$.

Proof. Let $\sigma \in S_n$, $\sigma \neq e$, and without loss of generality, assume that $C = (1, 2, ..., l)$ is a cycle of σ of largest length $l \geqslant 2$. We will exhibit tabloids $s, t \in T(\lambda)$ that belong to distinct proper cycles of σ^*.

Assume first that $\lambda_1 \geqslant 2$. Let s be a tabloid satisfying $\{1, 3\} \subseteq s[1]$ and $2 \in s[2]$.

If $l \geqslant 3$ then choose $t \in T(\lambda)$ such that $t[1]$ consists of the smallest elements in $[n] \backslash \{l-1, l\}$, and $\{l-1, l\} \subseteq t[2] \cup t[3]$. Now, $2 \in s[2] \cap \sigma^*(s)[1]$, so $\sigma^*(s) \neq s$ and s belongs to a proper cycle of σ^*. Similarly, $1 \in t[1] \backslash \sigma^*(t)[1]$, so t belongs to a proper cycle of σ^* as well. By construction, $t[1] \cap C = [m]$ for some $m \leqslant l-2$. Consider the tabloid $(\sigma^*)^i (t)$ in the cycle of t. Reducing i modulo l if necessary, assume $0 \leqslant i < l$. We need to show that $s \neq (\sigma^*)^i (t)$. This is true for $i = 0$, since it is easy to see that $s \neq t$. For $i = 1, ..., l-m$, we have $(\sigma^*)^i (t)[1] \cap C = \sigma^i([m])$, so $1 \in s[1] \backslash (\sigma^*)^i (t)[1]$. For $i = l+1-m \geqslant 3$, we have $3 \in s[1] \backslash (\sigma^*)^i (t)[1]$. Finally, if $l+2-m \leqslant i < l$, we have $2 \in (\sigma^*)^i (t)[1] \cap s[2]$. We conclude that s is not in the cycle of t, so σ^* has at least two proper cycles.

If $l = 2$ then choose $t \in T(\lambda)$ such that $1 \in t[1]$ and $\{2, 3\} \subseteq t[2] \cup t[3]$.

If, however, $\lambda_1 = 1$ (so $\lambda = (1, 1, ..., 1)$), choose $s, t \in T(\lambda)$ such that $1 \in s[1] \cap t[1]$, $2 \in s[2] \cap t[3]$, and $3 \in s[3] \cap t[2]$.

It is left for the reader to verify that s and t belong to different proper cycles of σ^* in these cases as well.

Thus, given any $\bar{\sigma}^*, \bar{\tau}^* \in V(\lambda)$, we have that, in $S_{T(\lambda)}$, $(\sigma^*)^{-1} \tau^* = (\sigma^{-1}\tau)^*$ is either the identity or has more than one proper cycle, and so, by Proposition 3.3, $\{\bar{\sigma}^*, \bar{\tau}^*\}$ is not a 1-face of $P(S_{T(\lambda)})$. ∎

We conclude that, in general, the graph of the assignment polytope $P(S_n)$ contains large stable sets. In particular, we have the following.

COROLLARY 3.5. *Let $\alpha(n) = \alpha(P(S_n))$ denote the stability number of the nth assignment polytope. Then $\alpha(n) = 2^{\Omega(\sqrt{n} \log n)}$.*

Proof. Given $n \in \mathbb{N}$, $n \geqslant 4$, let d be the largest integer such that $D = \binom{d}{2} \leqslant n$, and let $H = S_{[D]} \times S_{\{D+1\}} \times S_{\{D+2\}} \times \cdots \times S_{\{n\}}$. Identify $T((d-2, 2))$

with $[D]$, say, by sending a tabloid t with $t[2]=\{i,j\}$ $(i<j)$ to

$$d(i-1)-(i/2)(i+1)+j\in[D].$$

Then, the $(d-2,2)$ Young representation turns S_d into a subgroup S_d^* of $S_{[D]}$, and by the theorem, the set

$$STABLE_n=\{\bar{\sigma}\colon \sigma\in S_d^*\times S_{\{D+1\}}\times S_{\{D+2\}}\times\cdots\times S_{\{n\}}\}\subset \mathrm{mat}(n,\mathbb{R})$$

is stable in $\mathrm{conv}\{\bar{\sigma}\colon \sigma\in H\}\subseteq P(S_n)$, and so is stable in $P(S_n)$ as well. Thus, $\alpha(n)\geqslant d!\geqslant\sqrt{n}!=2^{\Omega(\sqrt{n}\log n)}$. ∎

Thus, for every $n\geqslant 4$ we have the set $STABLE_n$ of $n\times n$ permutation matrices, which is stable in $P(S_n)$. Let $STABLE=\bigcup_{n\geqslant 4} STABLE_n$. We now show that $STABLE$ could be efficiently decided. First, consider the following question. Given a partition $\lambda=(\lambda_1,\ldots,\lambda_k)\vdash n$ with $n\geqslant 2\lambda_k$ and a permutation $\sigma\in S_{T(\lambda)}$, is σ induced by the λ-Young representation, i.e., is $\sigma=\tau^*$ for some $\tau\in S_n$? The following procedure decides this question. Construct a candidate τ as follows. For each $i\in[n]$, choose two disjoint (λ_k-1)-subsets $S,T\subset[n]\backslash\{i\}$, and tabloids $s,t\in T(\lambda)$ such that $s[k]=S\cup\{i\}$ and $t[k]=T\cup\{i\}$. If $|\sigma(s)[k]\cap\sigma(t)[k]|\neq 1$ then σ is not induced. If $\sigma(s)[k]\cap\sigma(t)[k]=\{j\}$ then let $\tau(i)=j$. If τ was constructed successfully and is a permutation, then σ is induced if and only if $\sigma=\tau^*$. Using this procedure for the special case $\lambda=(n-2,2)$, we get the following.

Proposition 3.6. *The set STABLE is decidable in polynomial time.*

Proof. Given $A\in\mathrm{mat}(n,\mathbb{R})$, one finds a maximal d such that $D=\binom{d}{2}\leqslant n$, and verifies that A is a permutation matrix having the form

$$A=\begin{bmatrix} B & 0\\ 0 & I_{n-D}\end{bmatrix},$$

where I_{n-D} is the identity in $\mathrm{mat}(n-D,\mathbb{R})$. Applying the procedure described above, one verifies that the permutation in $S_{T((d-2,2))}=S_D$ corresponding to the permutation matrix $B\in\mathrm{mat}(D,\mathbb{R})$ is induced from some permutation in S_d. The matrix A is in $STABLE$ if and only if all verifications are true, and clearly this task can be performed in time polynomial in size(A), in fact, polynomial in n. ∎

4. Neighborliness Degree of Young Polytopes

In this section we establish lower and upper bounds on the neighborliness degree of Young polytopes, in terms of their defining partition λ.

Given a finite set A and a permutation $\sigma\in S_A$, it will be convenient to denote by $\mathrm{move}(\sigma)=\{a\in A\colon \sigma(a)\neq a\}$ the subset of A moved by σ.

LEMMA 4.1. *Let $\lambda \vdash n$ be any partition, and l such that $1 \leqslant l < n!$. If for every l distinct permutations other than the identity, $\sigma_1, ..., \sigma_l \in S_n \backslash \{e\}$, there exists a tabloid $t \in T(\lambda)$ such that $\sigma_i^*(t) \neq t$ $(i = 1, ..., l)$, then $P(\lambda)$ is l-neighborly.*

Proof. Let $F = \{\sigma_1, ..., \sigma_l\} \subseteq S_n$, and $\bar{F} = \{\bar{\sigma}^*: \sigma \in F\} \subseteq \text{ext}(P(\lambda))$. Let $\tau \in S_n \backslash F$. Then $e \notin \{\sigma\tau^{-1}: \sigma \in F\}$, so by the hypothesis there exists a tabloid t such that $(\sigma_i \tau^{-1})^* (t) \neq t$ $(i = 1, ..., l)$. Let $s = (\tau^{-1})^* (t)$. Then, $\tau^*(s) = \tau^*(\tau^{-1})^* (t) = (\tau\tau^{-1})^* (t) = t$, yet $\sigma_i^*(s) = \sigma_i^*(\tau^{-1})^* (s) = (\sigma_i \tau^{-1})^* (t) \neq t$ $(i = 1, ... l)$. Thus, we have $\bar{\tau}^*_{t,s} = 1$ while $(\bar{\sigma}_i^*)_{t,s} = 0$ $(i = 1, ..., l)$. Hence $\text{supp}(\bar{\tau}^*) \nsubseteq \text{supp}(\bar{F})$. Since τ was an arbitrary permutation not in F, we conclude from Proposition 3.2 that $\bar{F}$ is a face of $P(\lambda)$, and since F was an arbitrary l-subset of S_n, it follows that $P(\lambda)$ is indeed l-neighborly. ∎

We conclude our first lower bound.

COROLLARY 4.2. *If $\lambda \vdash n$ is a partition with $k+1$ parts $(k \geqslant 1)$ and $\lambda_1 \geqslant k$, then $P(\lambda)$ is k-neighborly.*

Proof. We prove that the hypothesis of Lemma 4.1 holds for $l = k$. Let $\sigma_1, ..., \sigma_k \in S_n \backslash \{e\}$ be k distinct permutations. Choose $j_i \in \text{move}(\sigma_i)$ $(i = 1, ..., k)$, and denote $S = \{j_1, ..., j_k\} = \{s_1, ..., s_m\}$ for some $m \leqslant k$ and distinct $s_1, ..., s_m$. Now, let $t \in T(\lambda)$ be a tabloid satisfying $s_i \in t[i+1]$ $(i = 1, ..., m)$ and $\{\sigma_1(j_1), ..., \sigma_k(j_k)\} \backslash S \subseteq t[1]$. It is clear that such a tabloid exists and that $\sigma_i^*(t) \neq t$ $(i = 1, ..., k)$. The corollary follows from Lemma 4.1. ∎

Behind this proof lies the more general observation that the hypothesis of Lemma 4.1 holds for a partition λ and a positive integer l if and only if a certain graph theoretical property on vertex coloring holds for λ and l. Recall that, given a (simple) graph $H = (V, E)$, a function $\chi: V \to [k]$ is a *k-coloring* of H if $\{u, v\} \in E$ implies $\chi(u) \neq \chi(v)$. Given $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0$, we say that $\chi: V \to [k]$ is a *λ-coloring* if it is a k-coloring and $|\chi^{-1}(i)| \leqslant \lambda_i$ $(i = 1, ..., k)$. The sufficient condition for $P(\lambda)$ to be l-neighborly given in Lemma 4.1, translates to a statement on λ-colorings.

LEMMA 4.3. *Let $\lambda = (\lambda_1, ..., \lambda_k) \vdash n$ be any partition, and l such that $1 \leqslant l \leqslant n/2$. For every l distinct permutations $\sigma_1, ..., \sigma_l \in S_n \backslash \{e\}$ there exists a tabloid $t \in T(\lambda)$ such that $\sigma_i^*(t) \neq t$ $(i = 1, ..., l)$, if and only if every graph with l edges admits a λ-coloring.*

Proof. Given distinct permutations $\sigma_1, ..., \sigma_l \in S_n \backslash \{e\}$, choose $j_i \in \text{move}(\sigma_i)$ $(i = 1, ..., l)$, and define a graph $H = (V, E)$ with

$$V = \{j_1, ..., j_l\} \cup \{\sigma_1(j_1), ..., \sigma_l(j_l)\}, \quad E = \{\{j_1, \sigma_1(j_1)\}, ..., \{j_l, \sigma_l(j_l)\}\}.$$

Since $|E| \leqslant l$, there exists a λ-coloring χ of H. Choose a tabloid $t \in T(\lambda)$ such that $\chi^{-1}(i) \subseteq t[i]$ $(i=1, ..., k)$. It is clear that $\sigma_i^*(t) \neq t$ $(i=1, ..., l)$. Conversely, consider any graph $H=(V, E)$ with l edges. Its set of vertices can be taken as $V=[m]$ for some $m \leqslant 2l \leqslant n$. For $i=1, ..., l$, define from the ith edge $f_i=\{u_i, v_i\}$ a transposition $\sigma_i \doteq (u_i, v_i)$. Let $t \in T(\lambda)$ be such that $\sigma_i^*(t) \neq t$ $(i=1, ..., l)$, and let $\chi: [m] \to [k]$ be the unique function satisfying $\chi^{-1}(i)=[m] \cap t[i]$ $(i=1, ..., k)$. It is clear that χ is a λ-coloring of H. ∎

The following graph theoretical proposition is straightforward [4, Corollary 1, p. 336].

PROPOSITION 4.4. *If at most p vertices in a graph H have degree at least p, then H admits a p-coloring.*

Proof. Let H have the set of vertices $[m]$, let the degree of i be denoted by d_i, and assume $d_1 \geqslant d_2 \cdots \geqslant d_m$. For $i=1, ..., p$, set $\chi(i)=i$, and for $i>p$ choose $\chi(i) \in [p]$ which had not been assigned to any of the (at most $p-1$) neighbors of i already colored. ∎

A second lower bound follows.

COROLLARY 4.5. *Let l be a positive integer, and $\lambda \vdash n$ be any partition with $k \geqslant \sqrt{2l}$ parts such that $\lambda_1 \geqslant \cdots \geqslant \lambda_{\lceil \sqrt{2l} \rceil} \geqslant 2l$. Then $P(\lambda)$ is l-neighborly.*

Proof. Any graph with l edges satisfies the hypothesis of Proposition 4.4 with $p=\lceil \sqrt{2l} \rceil \leqslant k$, and has at most $2l$ vertices. It therefore has a p-coloring χ, and $|\chi^{-1}(i)| \leqslant 2l \leqslant \lambda_i$ $(i=1, ..., p)$, so χ is a λ-coloring as well. The claim follows from Lemma 4.3 and Lemma 4.1. ∎

Next, we give an upper bound on the neighborliness degree of Young polytopes.

LEMMA 4.6. *Let $\lambda \vdash n$ be a partition with $k \geqslant 2$ parts such that $n>k$. Let H be the subgroup*

$$H=\{\sigma \in S_n : \mathrm{move}(\sigma) \subseteq [k+1]\} = S_{[k+1]} \times S_{\{k+2\}} \times S_{\{k+3\}} \times \cdots \times S_{\{n\}},$$

and let S be the subgroup of the even permutations in H.

Then the $\frac{1}{2}(k+1)!$-set $\bar{S}^* = \{\bar{\sigma}^* : \sigma \in S\} \subseteq \mathrm{ext}(P(\lambda))$ is not a face of $P(\lambda)$.

Proof. Let $T=H \backslash S$ be the set of odd permutations in H, and let $\bar{T}^* = \{\bar{\sigma}^* : \sigma \in T\}$. We will show that $\sum \{\bar{\sigma}^* : \sigma \in S\} = \sum \{\bar{\sigma}^* : \sigma \in T\}$, which, since $|S|=|T|$, shows that $\mathrm{conv}(\bar{S}^*) \cap \mathrm{conv}(\bar{T}^*) \neq \varnothing$, so $\bar{S}^*$ is not a face. So, we have to show that, for any two tabloids $t, s \in T(\lambda)$, we have $\sum \{\bar{\sigma}^*_{t,s} : \sigma \in S\} = \sum \{\bar{\sigma}^*_{s,t} : \sigma \in T\}$. Since $\bar{\sigma}^*_{t,s}=1$ if $\sigma^*(t)=s$ and is zero otherwise, this amounts to showing that $|C_{s,t} \cap S| = |C_{s,t} \cap T|$, where $C_{s,t} = \{\sigma \in S_n : \sigma^*(t) = s\}$. Now $C_{s,t}$ is a left coset of the *row stabilizer*

$$R_t = \{\sigma \in S_n : \sigma^*(t) = t\} = S_{t[1]} \times S_{t[2]} \times \cdots \times S_{t[k]}$$

of t, say $C_{s,t} = \sigma R_t$.

It suffices to show, then, that the set $R_t \cap \sigma^{-1}H$ contains the same number of odd and even permutations. Let $\pi \in \sigma^{-1}H$. Then for $i \geqslant k+2$ we have $\pi(i) = \sigma^{-1}(i)$, and $\pi([k+1]) = \{\sigma^{-1}(1), ..., \sigma^{-1}(k+1)\}$. Let

$$I_i = t[i] \cap [k+1], \qquad J_i = t[i] \cap \{\sigma^{-1}(1), ..., \sigma^{-1}(k+1)\} \ (i = 1, ..., k).$$

For $\pi \in \sigma^{-1}H$ to be in R_t, both i and $\sigma^{-1}(i)$ must be in the same row of t for $i \geqslant k+2$, and we must have $|I_j| = |J_j|$ $(j = 1, ..., k)$. If this is not the case, then $R_t \cap \sigma^{-1}H = \varnothing$ and we are trivially done, so assume the tabloid t satisfies these conditions. Then, the set $R_t \cap \sigma^{-1}H$ consists exactly of those permutations π satisfying $\pi(i) = \sigma^{-1}(i)$ $(i \geqslant k+2)$ and $\pi(I_i) = J_i$ $(i = 1, ..., k)$, and is therefore a left coset of the subgroup

$$K = S_{I_1} \times \cdots \times S_{I_k} \times S_{\{k+2\}} \times S_{\{k+3\}} \times \cdots \times S_{\{n\}}.$$

Since $\bigcup_{i=1}^{k} I_i = [k+1]$, there exists some I_m such that $|I_m| \geqslant 2$, and so it is clear that K contains the same number of even and odd permutations. This completes the proof. ▌

Combining Corollary 4.5 and Lemma 4.6, we have the following.

THEOREM 4.7. *Let $\lambda = (\lambda_1, ..., \lambda_k)$ be such that $k \geqslant 2$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant k^2$. Let l denote the largest positive integer for which the Young polytope $P(\lambda)$ is l-neighborly. Then $\lfloor k^2/2 \rfloor \leqslant l < \frac{1}{2}(k+1)!$.*

It is very probable that the set $\{(\lambda, l): P(\lambda)$ is l-neighborly$\}$ does not admit an efficient decision procedure and a simple characterization. It is not even clear if it belongs to the computational complexity class *NP* or its complement *co − NP*. However, for a fixed k and $\lambda_2, ..., \lambda_k$, the following holds.

PROPOSITION 4.8. *Let $k \geqslant 2$ and $\lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$, and for $n \geqslant \lambda_2 + \sum_{i=2}^{k} \lambda_i$, let $\lambda(n) = (n - \sum_{i=2}^{k} \lambda_i, \lambda_2, ..., \lambda_k)$. Then the set*

$$A = \{(n, l): P(\lambda(n)) \text{ is } l\text{-neighborly}\}$$

is in co − NP.

Proof. The crucial observation is that the affine dimension of $P(\lambda(n))$ is bounded from above by $d(n)^2$, where $d(n) = \binom{n}{\lambda} \leqslant n^{(\sum_{i=2}^{k} \lambda_i)}$. For a pair $(n, l) \notin A$, there exists an l-subset $F \subset S_n$ such that $\bar{F} = \{\bar{\sigma}^* : \sigma \in F\}$ is not a face of $P(\lambda(n))$. Then, by Proposition 3.1, there exist $S, T \subset S_n$ such that $S \subseteq F$, $T \nsubseteq F$, and $(\bar{S}, \bar{T})$ is a minimal Radon partition, where

$\bar{S}=\{\bar{\sigma}^*: \sigma \in S\}$ and $\bar{T}=\{\bar{\sigma}^*: \sigma \in T\}$. By definition of a minimal Radon partition, $|S \cup T| \leqslant d(n)^2+2$. Also, a matrix $M \in \mathrm{mat}(d(n), \mathbb{R})$ exists such that size(M) is bounded by a polynomial function in $d(n)$ and $M \in \mathrm{conv}(\bar{S}) \cap \mathrm{conv}(\bar{T})$, since the matrices in $\bar{S}, \bar{T}$ are $\{0, 1\}$-valued (cf. [23]). Thus, given $F, S, T \subseteq S_n$ and $M \in \mathrm{mat}(d(n), \mathbb{R})$ as above, one can verify that $S \subseteq F$ and $T \nsubseteq F$, construct the corresponding sets of matrices $\bar{S}, \bar{T} \subseteq \mathrm{mat}(d(n), \mathbb{R})$, and check that $M \in \mathrm{conv}(\bar{S}) \cap \mathrm{conv}(\bar{T})$ in time polynomial in $n+l$.

Finally, we characterize those partitions $\lambda \vdash n$ for which $P(\lambda)$ is 2-neighborly.

PROPOSITION 4.9. *For any partition $\lambda \vdash n$ other than (n) and $(n-1, 1)$, the Young polytope $P(\lambda)$ is 2-neighborly.*

Proof. The polytope $P((1, 1, 1))$ being a simplex, we may assume $n \geqslant 4$. By Lemma 4.3, it is enough to exhibit λ-colorings χ_1 and χ_2 for the only two nonisomorphic simple graphs with two edges, $H_1=([3], \{\{1, 2\}, \{1, 3\}\})$ and $H_2=([4], \{\{1, 2\}, \{3, 4\}\})$, respectively. If $\lambda_1=1$, then set $\chi_1(i)=i$ $(i=1, 2, 3)$ and $\chi_2(i)=i$ $(i=1, ..., 4)$. If $\lambda_1 \geqslant 2$, set $\chi_i(1)=2$, $\chi_i(2)=\chi_i(3)=1$ $(i=1, 2)$, and, if $\lambda_2=1$ set $\chi_2(4)=3$ whereas if $\lambda_2 \geqslant 2$ set $\chi_2(4)=2$. ∎

We conclude in particular that for $n \geqslant 4$ the graph of $P((n-2, 2))$ is a clique, which is interesting in two respects. First, it provides an example of a sequence of 2-neighborly polytopes on which the linear optimization related decision problem is *NP*-complete. Second, while the adjacency relation is trivial for this sequence, it is *NP*-complete for the projected sequence $P((n-2, 2), h_n)$ of symmetric traveling salesman polytopes [21].

5. ORBIT STRATIFICATIONS AND A PROBLEM OF KOZEN

We now fix an arbitrary subgroup G of S_d. For any point $x \in \mathbb{R}^d$ we have its orbit $G \cdot x=\{\pi(\sigma)(x): \sigma \in G\}$ under the standard representation $\pi: S_d \rightarrow GL(\mathbb{R}^d)$. With each $x \in \mathbb{R}^d$ we associate the polytope $P(G, x)=\mathrm{conv}(G \cdot x)$ as before, and the matroid $M(G, x)$ and oriented matroid $O(G, x)$ defined by affine dependencies on the orbit $G \cdot x$. In order to simplify the discussion, we concentrate on the set $\mathscr{D}(\mathbb{R}_+^d)$ of points in the nonnegative orthant with pairwise distinct coordinates. The orbit of any $x \in \mathscr{D}(\mathbb{R}_+^d)$, which is the underlying set of any of the three structures above, can then be indexed simply by the group G. We say that two polytopes P and Q, having their vertices indexed by G, are (combinatorially) *isomorphic*, written $P \simeq Q$, if there exists a bijection $\varphi: G \to G$ such that a subset $F \subseteq G$ is a face of P if and only if $\varphi(F) = \{\varphi(\sigma): \sigma \in F\}$ is a face of Q. If this bijection is the identity on G, we use the equality sign $P = Q$. Matroid and oriented matroid isomorphisms are defined in a similar manner.

For $x \in \mathscr{D}(\mathbb{R}^d_+)$, we observe that the orbit $G \cdot x$ lies on an affine hyperplane in $\mathbb{R}^d$, so $M(G, x)$ is also the matroid of *linear* dependencies on the orbit, and similarly for the oriented matroid $O(G, x)$.

The isomorphism type of each of these three structures induces a stratification of affine space $\mathbb{R}^d$. Specifically, two points $x, y \in \mathbb{R}^d$ are in the same *polytope stratum* induced by G if $P(G, x) \simeq P(G, y)$, and are in the same *matroid stratum* (respectively, *oriented matroid stratum*) if $M(G, x) \simeq M(G, y)$ (respectively, $O(G, x) \simeq O(G, y)$). Note that the oriented matroid stratification is a refinement of both the other two (the face lattice could be read off from the oriented matroid, cf. [16]). One can also consider the stratifications induced by the labeled isomorphism type of each of the three structures.

We say that a point $x = (x_1, ..., x_d) \in \mathbb{R}^d$ is *generic*, if $x_1, ..., x_d \in \mathbb{R}$ are algebraically independent over the field $\mathbb{Q}$ of rational numbers. The following motivating question was raised by D. Kozen. Is it true that any two generic points $x, y \in \mathbb{R}^d$ lie in the same polytope stratum? It will become evident below that it is natural to restrict attention to generic points.

Let $\mathscr{K} = \mathbb{Q}(X_1, ..., X_d)$ be the field of rational functions on the variables $X_1, ..., X_d$ with rational coefficients. Define the *generic matroid* $M(G)$ of the subgroup G as the matroid of $\mathscr{K}$-linear dependencies on the set of points $\{(X_{\sigma(1)}, ..., X_{\sigma(d)}): \sigma \in G\} \subseteq \mathscr{K}^d$, indexed, as the orbit matroids, by G. Given a point $x = (x_1, ..., x_d) \in \mathscr{D}(\mathbb{R}^d_+)$, the specialization map $\varphi: \mathscr{K} \to \mathbb{Q}(x_1, ..., x_d)$: $X_i \mapsto x_i$ induces a weak map of matroids (cf. [15]). This can be rephrased as follows.

PROPOSITION 5.1. *For any point $x \in \mathscr{D}(\mathbb{R}^d_+)$, the orbit matroid $M(G, x)$ is a* weak image *of the generic matroid $M(G)$, i.e., if $F \subseteq G$ is an independent set of $M(G, x)$, then it is also an independent set of $M(G)$.*

The following corollary is useful in deciding when the orbit matroid of a point $x \in \mathbb{R}^d$ is the generic matroid.

COROLLARY 5.2. *Given a point $x \in \mathscr{D}(\mathbb{R}^d_+)$, if $M(G, x)$ and $M(G)$ have the same number of bases, then $M(G, x) = M(G)$.*

If $x \in \mathbb{R}^d$ is a generic point, then the specialization map above is a field isomorphism. Hence, the induced map of matroids is an isomorphism, yielding the following.

PROPOSITION 5.3. *For any generic point $x \in \mathbb{R}^d$ we have $M(G, x) = M(G)$.*

Thus, all generic points lie in the same matroid stratum, and the motivation for considering generic points becomes apparent.

We need to recall a few facts about oriented matroids (cf. [7]). Let O be an oriented matroid of rank r, defined on a finite set G. Its *chirotope* is the signed base map $\chi_0: \binom{G}{r} \to \{-1, 0, 1\}$ on the collection of r-subsets of G, which entirely determines the oriented matroid. In particular, an r-set $F \subseteq G$ is a basis of O is and only if $\chi_O(F) \neq 0$. Now, for $x \in \mathscr{D}(\mathbb{R}^d_+)$ and a d-set $F \subseteq G \subseteq S_d$, let $F(x) \in \mathrm{mat}(d, \mathbb{R})$ be the matrix having column set $\{(x_{\sigma(1)}, ..., x_{\sigma(d)}): \sigma \in F\}$. If $\mathrm{rank}(O(G, x)) = d$, then its chirotope is given by

$$\chi_x(F) = \chi_{O(G,x)}(F) = \mathrm{sign}(\det(F(x))).$$

Finally, we need the following.

Observation 5.4. For any $x \in \mathbb{R}^d$, the orbit polytope $P(G, x)$ is G-symmetric, i.e., for any permutation $\sigma \in G$, the bijection $\sigma: G \to G: \tau \mapsto \sigma\tau$ gives an automorphism of $P(G, x)$. Similarly, $M(G, x)$ and $O(G, x)$ are G-symmetric.

This is true since the linear transformation $\pi(\sigma)$: $\mathbb{R}^d \to \mathbb{R}^d$ maps every orbit of G bijectively onto itself. Thus, each of the structures is entirely determined by the *vertex figure* of the identity $e \in G$. For example, the list of those subsets $F \subseteq G$ containing e which are faces of $P(G, x)$ determines the entire face lattice of $P(G, x)$.

We are now in a position to provide the negative answer to Kozen's question.

THEOREM 5.5. *There exists a subgroup G of S_6, isomorphic to S_4, and two generic points $x, y \in \mathbb{R}^6$ for which $M(G, x) = M(G, y)$, yet $P(G, x) \not\simeq P(G, y)$ and $O(G, x) \not\simeq O(G, y)$.*

Proof. Let $\lambda = (2, 2) \vdash 4$, and let $G = S_4^*$ be the subgroup of $S_{T(\lambda)}$ induced by the λ-Young representation of S_4. Identifying $T(\lambda)$ with $[6] = \{1, ..., 6\}$ by sending a tabloid t with $t[2] = \{i, j\}$ $(i < j)$ to $4(i-1) - (i/2)(i+1) + j \in [6]$, we turn G into a subgroup of S_6.

Computing symbolically with the computer program "Maple," it was verified for the generic matroid that $\mathrm{rank}(M(G)) = 6$, and that 679 of the 6-subsets of G containing the identity $e \in G$ are not bases of $M(G)$.

Similarly, for $u = (1, 10, 11, 30, 70, 90)$ and $v = (19, 5, 83, 29, 67, 37)$, it was verified that $\mathrm{rank}(M(G, u)) = \mathrm{rank}(M(G, v)) = 6$ and each of these two matroids has 679 nonbasic 6-subsets containing e as well.

It follows from Observation 5.4 and Corollary 5.2 that $M(G, u) = M(G) = M(G, v)$. Now, consider the chirotope χ_u of $O(G, u)$. If a 6-subset

$F \subseteq G$ is a basis of $M(G)$, it is also a basis of $M(G, u)$, so $\chi_u(F) \neq 0$, implying $\det(F(u)) \neq 0$. Thus, by continuity of the determinant function, and since the set of generic points is dense in $\mathbb{R}^6$, we can perturb u slightly to get a generic point $x \in \mathbb{R}^6$ such that, for any $F \in \binom{G}{6}$ which is a basis of $M(G)$, we have $\chi_x(F) = \text{sign}(\det(F(x))) = \text{sign}(\det(F(u))) = \chi_u(F)$. If $F \in \binom{G}{6}$ is not a basis of $M(G)$, then by Proposition 5.1 it is neither a basis of $M(G, u)$ nor of $M(G, x)$, so $\chi_x(F) = 0 = \chi_u(F)$. Thus, the chirotopes χ_x, χ_u are identical, showing that $O(G, x) = O(G, u)$. Similarly, we can perturb v to obtain a generic point $y \in \mathbb{R}^6$ such that $O(G, y) = O(G, v)$.

It follows that $P(G, x) = P(G, u)$ and $P(G, y) = P(G, v)$ as well.

Now, the facets of $P(G, u)$ and $P(G, v)$ containing the identity e were obtained. In $P(G, u)$ there are 4 such facets containing 8 vertices each, 17 facets containing 6 vertices each, and 20 facets containing 5 vertices each. Appealing to Observation 5.4 and recalling $|G| = 24$, by counting vertex-facet incidences separately for facets containing 8, 6, or 5 vertices each, we get that $P(G,u)$ has $(24 \cdot 4)/8 + (24 \cdot 17)/6 + (24 \cdot 20)/5 = 176$ facets.

Similarly, $P(G, v)$ has 6 such facets containing 8 vertices each, 8 facets containing 6 vertices each, and 5 facets containing 5 vertices each, and it turns out to have 74 facets.

Thus, $P(G, x) \not\simeq P(G, y)$ and hence also $O(G, x) \not\simeq O(G, y)$ and, furthermore, even the number of faces of the same dimension of $P(G\, x)$ and $P(G, y)$ are not the same. ∎

Thus, in general, both the polytope and oriented matroid stratifications of $\mathbb{R}^d$ induced by G are nontrivial on the set of generic points.

We remark that the (2,2)-Young representation $\pi_{(2,2)}\colon S_4 \to S_6$ is the direct sum of three mutually nonisomorphic irreducible representations of S_4, so our example shows that these stratifications are nontrivial even if the underlying representation is multiplicity-free.

6. Discussion

This report raises many questions, and much work is yet to be done. It would be nice to get closer to a complete characterization of pairs (λ, l) for which $P(\lambda)$ is l-neighborly. Surely, using the sufficient condition derived through Proposition 3.2 and Lemmas 4.1 and 4.3 would not be enough. Yet, even getting the most out of this condition, namely, characterizing those pairs (λ, l) for which every graph with l edges is λ-colorable, seems a challenging graph theoretical question. A first step might be to consider the computational complexity of the set $\{(\lambda, l)\colon P(\lambda) \text{ is } l\text{-neighborly}\}$, which at first glance seems to be neither in NP nor in $co-NP$.

The polytope stratifications induced by Young representations are far from being understood. It is particularly interesting to consider sequences of polytope fibers for which the associated decision problems have different computational complexities, arising from the same sequence of polytope bundles, such as $P((n-2, 2), m_n)$ and $P((n-2, 2), h_n)$, $n \geqslant 4$. It would be very interesting to better understand the way in which the polytopes in the first, tractable sequence, continuously deform into the corresponding polytopes in the second, intractable sequence, when following a uniformly specified sequence of paths $p_n: [0, 1] \to [m_n, h_n]$ in the base spaces.

Another interesting direction is to investigate the generic matroid $M(G)$ associated with each subgroup G of S_n.

The general problem is the study of the representation polytope $P(\rho) = \operatorname{conv}(\{\rho(g): g \in G\})$ and the orbit polytopes $P(\rho, x) = \operatorname{conv}(\{\rho(g)(x): g \in G\})$, $x \in \mathbb{R}^d$, where $\rho: G \to GL(\mathbb{R}^d)$ is any real representation of any finite group G.

We have shown that, in order for such a representation to have a trivial polytope stratification on the set of generic points, being multiplicity free is not enough. The next question is, obviously, whether or not the representation being *irreducible* does suffice.

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